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Symmetries and observables for BF theories in superspace

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Abstract. The supersymmetric version of a topological quantum field theory describing flat connections, the super BF theory, is studied in the superspace formalism. A set of observables related to topological invariants is derived from the curvature of the superspace. Analogously to the non-supersymmetric versions the theory exhibits a vector-like supersymmetry. The role of the vector supersymmetry and an additional new symmetry of the action in the construction of observables is explained.

1. Introduction

Topological field theories offer an intriguing possibility to combine ideas from physics and mathematics. They are quantum field theories with no physical degrees of freedom and their properties are fully determined by the global structure of the manifold they are defined on. A remarkable feature is that for many topological theories, like the Donaldson theory [1] and Chern–Simons (CS) theory, the expectation values of the observables are topological invariants.

The Chern–Simons theory provides a three-dimensional interpretation of the theory of knots: the correlators of its observables, Wilson loops, are related to the Jones polynomials of knot theory [2]. Another important application of CS theory is $(2+1)$-dimensional gravity. CS action with Poincaré group as the gauge group is the Einstein–Hilbert action [3], giving a gauge theory interpretation of gravity in $(2+1)$ dimensions. However, the Chern–Simons theory is defined only in three dimensions. The generalization to arbitrary dimensions [4, 5] are called BF models or antisymmetric tensor models. They, like the CS theory, describe the moduli space of flat connections and their observables are related to the linking and intersection numbers of manifolds. The supersymmetric BF theories (SBF) were introduced in [6] as a supersymmetric version of $(2+1)$-dimensional topological gravity. There it was also shown that the partition function of three-dimensional SBF computes a topological invariant, the Casson invariant. Generalizations of SBF to other dimensions were considered in [7] and [5, 8].

In this paper we study supersymmetric BF models. We are particularly interested in finding new observables and possible topological invariants for $3d$ SBF theories, besides the partition function. By formulating the theory in superspace a large set of observables, including previously unknown ones, can be derived from the superspace curvature. In [9] a vector-like supersymmetry similar to that found in ordinary BF models and Chern–Simons theory [10, 11] was constructed for the SBF models. In particular, the hierarchy of observables constructed from the supercurvature can be derived from one initial observable

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with the help of the vector supersymmetry. Using the superspace formulation we extend this construction to include also the anti-BRST and anti-vector supersymmetries, in addition to the usual BRST and vector supersymmetries.

This paper is organized as follows: in section 2 we will introduce the model and write it in the superspace. It turns out that in the superspace formalism many features of the CS theory can be generalized directly to SBF theory. In section 3 we derive the set of observables and discuss their relation to topological invariants. In section 4 we generalize the vector supersymmetry to SBF and show how it can be used to construct new observables.

2. Supersymmetric BF theories

The classical action or non-supersymmetric BF model in d dimensions is

$$S_0 = \int \mathrm{d}^d x \, B_n^0 F_A \tag{1}$$

where B_n^0 is a $n = d - 2$ form (with ghost number zero) and F_A is the curvature 2-form $F_A = \mathrm{d}A + \frac{1}{2}[A, A]$. In addition to the normal Yang–Mills gauge symmetry $A \to A + \mathrm{d}_A\omega_0$, the action is invariant under the transformation $B_n \to B_n + \mathrm{d}_A\omega_{n-1}$ caused by the Bianchi identity. In dimensions higher than three this symmetry is reducible:

$$\omega_{n-1} \to \mathrm{d}_A\omega_{n-2} \qquad \text{etc}$$

and additional ghost fields are needed in order to fix the gauge according to the Batalin–Vilkovisky procedure.

In three dimensions the BF theory is closely related to Chern–Simons theory: the CS theory for the tangent group $TG \simeq (G, \underline{g})$ is equivalent to the BF theory for G [12]. In TG the Chern–Simons connection 1-form splits into two parts A and B, the basic fields of the BF theory. This makes it possible to construct the classical action of BF theories, find the BRST transformations and fix the gauge easily by studying the CS theory for the tangent group.

For the supersymmetric extension of the three-dimensional BF model the situation is quite similar—the action and many properties of the theory can be expressed in terms of super CS theory. This is done elegantly by formulating the theory in superspace with two anticommuting Grassmannian coordinates θ, $\bar{\theta}$ in addition to the normal spacetime coordinates x_μ. Here we will mainly concentrate in the three-dimensional case but with slight modifications the method is suited for SBF models in other dimensions.

The integration over the Grassmannian variables is normalized as

$$\int \mathrm{d}\bar{\theta}\,\mathrm{d}\theta \left\{ \begin{array}{c} 1 \\ \theta \\ \bar{\theta} \\ \theta\bar{\theta} \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right\}. \tag{2}$$

If the coordinates θ and $\bar{\theta}$ are associated with ghost numbers -1 and 1 the superspace connection 1-form $\hat{\mathcal{A}}$ in $(3 + 2)$ dimensions is written†

$$\hat{\mathcal{A}} = \hat{A}_\mu^0 \,\mathrm{d}x^\mu + \hat{A}_\theta^1 \,\mathrm{d}\theta + \hat{A}_{\bar{\theta}}^{-1} \,\mathrm{d}\bar{\theta} \tag{3}$$

† Note that we will use graded differential forms X_p^q with ordinary form degree p and ghost number q. Two graded forms satisfy $X_p^q Y_s^r = (-1)^{(q+p)(r+s)} Y_s^r X_p^q$. All the commutators should also be considered as graded.

where the superfields $\hat{A}^0_\mu\,\mathrm{d}x^\mu$, $\hat{A}^1_\theta$ and $\hat{A}^{-1}_{\bar\theta}$ can be further expanded as

$$\begin{aligned}
\hat{A}^0_\mu &= A_\mu - \theta\psi_\mu + \bar\theta\chi_\mu + \theta\bar\theta B_\mu \\
\hat{A}^1_\theta &= c - \theta\phi + \bar\theta\rho + \theta\bar\theta\eta \\
\hat{A}^{-1}_{\bar\theta} &= \bar\eta - \theta\bar\rho - \bar\theta\bar\phi + \theta\bar\theta\bar c.
\end{aligned}$$

The components can be identified with the fields of three-dimensional super BF theory: ψ^1_μ and χ^{-1}_μ are the superpartners of the connection A^0_μ and field B^0_μ, while $\rho^0_0, \bar\rho^0_0$ and $\phi^2_0, \bar\phi^{-2}_0$ are their corresponding ghosts and antighosts. With these definitions the classical action of the SBF model can be written as the action of the super CS theory:

$$S_{cl} = \int \mathrm{d}^3x (BF_A - \chi\,\mathrm{d}_A\psi) = \frac{1}{2}\int \mathrm{d}^3x\,\mathrm{d}^2\theta(\hat{\mathcal{A}}\hat{\mathrm{d}}\hat{\mathcal{A}} + \tfrac{2}{3}\hat{\mathcal{A}}[\hat{\mathcal{A}},\hat{\mathcal{A}}]). \tag{4}$$

To obtain the quantum action one has to fix the gauge symmetry $\hat{A}^0 \to \hat{A}^0 + \mathrm{d}_{\hat{A}^0}\omega$ by adding to the action a BRST exact gauge fixing term.

The BRST transformations of the fields can be derived from the superspace curvature 2-form using a method similar to that of [8, 9, 13–16] for Donaldson theory and Witten-type topological theories. However, because of the $N = 2$ superspace with two anticommuting coordinates of opposite ghost numbers we can extend this method to also include the anti-BRST symmetry. We define the superspace curvature as

$$\hat{\mathcal{F}} = (\mathrm{d}x^\mu\partial_\mu + \mathrm{d}\theta\delta + \mathrm{d}\bar\theta\bar\delta)\hat{\mathcal{A}} + \tfrac{1}{2}[\hat{\mathcal{A}},\hat{\mathcal{A}}] \tag{5}$$

and impose the ‘horizontality condition’

$$\hat{\mathcal{F}} \equiv \hat{F}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu - (\mathrm{d}\theta\partial_\theta + \mathrm{d}\bar\theta\partial_{\bar\theta})\hat{\mathcal{A}} \tag{6}$$

which truncates the curvature to the physical part independent of $\mathrm{d}\theta$ and $\mathrm{d}\bar\theta$ (and consequently of the ghost fields), and identifies the BRST operator δ with ∂_θ and $\bar\delta$ with $\partial_{\bar\theta}$. This gives the BRST transformations for the component fields:

$$\begin{aligned}
\delta A &= -\mathrm{d}_A c + \psi & \delta B &= -\mathrm{d}_A\eta - [c, B] + [\phi, \chi] + [\psi, \rho] \\
\delta c &= -\tfrac{1}{2}[c, c] + \phi & \delta\eta &= -[c, \eta] + [\phi, \rho] \\
\delta\psi &= -\mathrm{d}_A\phi - [c, \psi] & \delta\chi &= -\mathrm{d}_A\rho - [c, \chi] + B \\
\delta\phi &= -[c, \phi] & \delta\rho &= -[c, \rho] + \eta
\end{aligned} \tag{7}$$

which have to be supplemented with the transformations of the antighosts and Lagrange multipliers $\lambda^0_0, b^0_0, \beta^{-1}_0$ and σ^1_0 for the gauge fixing conditions of the fields A, B, ψ and χ. The Lagrange multipliers can be combined into a superfield

$$\hat{\Lambda}^0_0 = \lambda - \theta\sigma - \bar\theta\beta + \theta\bar\theta b. \tag{8}$$

The simplest choice for the BRST transformations would be

$$\delta\hat{\mathcal{A}}^{-1}_0 = -\hat\Lambda \qquad \delta\hat\Lambda = 0$$

but with suitable field redefinitions these can be put into a form which will be more convenient later:

$$\begin{aligned}
\delta\bar c &= -b & \delta\bar\eta &= -\lambda - [c, \bar\eta] + \bar\rho \\
\delta b &= 0 & \delta\lambda &= -[c, \lambda] - [\phi, \bar\eta] - \sigma \\
\delta\bar\phi &= \beta - \bar c & \delta\bar\rho &= \sigma - [c, \bar\rho] \\
\delta\beta &= -b & \delta\sigma &= -[c, \sigma] + [\phi, \bar\rho].
\end{aligned} \tag{9}$$

The gauge fixing part of the supersymmetric action is chosen to be

$$
\begin{aligned}
S_{gf} &= \int \mathrm{d}^2\theta\, \delta(\hat{A}_{\bar{\theta}}^{-1}\mathrm{d} * \hat{A}^0) \\
&= \int \mathrm{d}^3x\, (-b\mathrm{d} * A - \lambda\mathrm{d} * B + \beta\mathrm{d} * \psi + \sigma\mathrm{d} * \chi + \bar{c}\mathrm{d} * \mathrm{d}_A c + \bar{\eta}\mathrm{d} * \mathrm{d}_A\eta \\
&\quad + \bar{\phi}\mathrm{d} * \mathrm{d}_A\phi + \bar{\rho}\mathrm{d} * \mathrm{d}_A\rho - \bar{\eta}[\mathrm{d}c, *B] + \bar{\eta}\mathrm{d}[\phi, *\chi] \\
&\quad - \bar{\eta}\mathrm{d}[\rho, *\psi] + \bar{\rho}[\mathrm{d}c, *\chi] + \bar{\phi}\mathrm{d}[c, *\psi]).
\end{aligned}
\tag{10}
$$

Note also that unlike in the ordinary BF model the classical action (4) is now BRST exact:

$$S_{\mathrm{cl}} = \int \mathrm{d}^3x\, \delta(\chi F_A).$$

This shows that the supersymmetric BF model is a Witten-type topological theory with a δ-exact action, whereas the ordinary non-Abelian BF models are Schwartz-type theories [12].

The gauge fixing term $S_{\mathrm{gf}} = \int \mathrm{d}^2\theta\, \delta(\hat{A}_{\bar{\theta}}^{-1}\mathrm{d} * \hat{A}^0)$ is formally similar to that of Chern–Simons theory quantized in the Landau gauge $\mathrm{d} * \mathcal{A} = 0$:

$$S_{\mathrm{gf}}^{\mathrm{CS}} = \int \mathrm{d}^3x\, (\delta\bar{\mathcal{C}}\mathrm{d} * \mathcal{A}).$$

In CS theory the BRST and anti-BRST operators are related by the transformation obtained by integrating the quantum action by parts [17]

$$S_{\mathrm{q}}^{\mathrm{CS}} = \int \mathrm{d}^3x\, (\mathcal{A}\mathrm{d}\mathcal{A} + \tfrac{2}{3}\mathcal{A}[\mathcal{A}, \mathcal{A}] - \Lambda\mathrm{d} * \mathcal{A} - \bar{\mathcal{C}}\mathrm{d} * \mathrm{d}_{\mathcal{A}}\mathcal{C}).$$

The integrated action is equivalent to the original action after a change of fields which leaves the connection $\mathcal{A}$ unchanged but takes the ghosts $\mathcal{C}$ to the antighosts $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}}$ to $-\mathcal{C}$. The Lagrange multiplier field Λ transforms as $\Lambda \to \Lambda - [\mathcal{C}, \bar{\mathcal{C}}]$. This transformation of the fields maps δ to $\bar{\delta}$.

For the super CS and consequently for the three-dimensional SBF theory the situation is again analogous. Integrating the gauge fixed quantum action $S_{\mathrm{q}} = S_{\mathrm{cl}} + S_{\mathrm{gf}}$ (4) and (10) by parts we find the superspace version of the transformation which relates BRST and anti-BRST operators. In superspace language the transformation rules can be expressed compactly by demanding that under the 'conjugation' of the Grassmann variables

$$\theta \to \bar{\theta} \qquad \bar{\theta} \to -\theta \tag{11}$$

the total superspace connection $\hat{\mathcal{A}}$ stays the same while the operators change as $\delta \to \bar{\delta}$, $\bar{\delta} \to -\delta$. The transformations for the Lagrange multipliers are somewhat more complicated

$$
\begin{aligned}
&\lambda \to \lambda + [c, \bar{\eta}] \qquad && b \to b - [c, \bar{c}] - [\eta, \bar{\eta}] - [\rho, \bar{\rho}] - [\phi, \bar{\phi}] \\
&\sigma \to \beta + [c, \bar{\phi}] \qquad && \beta \to -\sigma + [\phi, \bar{\eta}].
\end{aligned}
\tag{12}
$$

For BF theories in dimensions other than three the situation is slightly more complicated because the A and B fields cannot be combined into one connection. In d dimensions B is a $(d-2)$-form and additional fields will be needed to take care of the reducibility. It is, however, possible to use truncated fields and write the components of $\hat{\mathcal{A}}$ as

$$
\begin{aligned}
&\hat{A}_\mu^0 = A_\mu - \theta\psi_\mu \\
&\hat{A}_\theta^1 = c - \theta\phi \\
&\hat{A}_{\bar{\theta}}^{-1} = -\bar{\theta}\bar{\phi} + \theta\bar{\theta}\bar{c}
\end{aligned}
\tag{13}
$$

and similarly for the $(d-2)$ superform $(\hat{\mathcal{B}})$, which now contains in addition to B, χ and their ghosts also the whole tower of ghosts for ghosts from the Batalin–Vilkovisky gauge fixing. The curvature of the B sector is defined as

$$\hat{\mathcal{R}} = (\mathrm{d}x^\mu\, \partial_\mu + \mathrm{d}\theta\, \delta)\hat{\mathcal{B}} + [\hat{\mathcal{A}}, \hat{\mathcal{B}}]. \tag{14}$$

It satisfies a Bianchi identity, and again after imposing the horizontality condition similar to (6) it reproduces the correct nilpotent BRST transformations. Since the A and B sectors do not appear symmetrically in the action there exists no partial integration symmetry and thus no anti-BRST operator $\bar{\delta}$.

3. Observables

In order to establish that the observables of the theory are indeed topological invariants it must be checked that they are BRST closed, their expectation values do not depend on variations of the metric and, if they are integrals of some local functionals, that their BRST cohomology depends only on the homology class of the integration contour. The partition function of the three-dimensional SBF model

$$Z_{3d} = \int \mathrm{e}^{\mathrm{i}S_{\mathrm{q}}}$$

obviously satisfies all the requirements and can be shown to equal the Casson invariant of the manifold [6, 18]. We will now derive a set of other observables for $3d$ SBF from the superspace curvature (5) and see if they too could correspond to topological invariants.

The Bianchi identity

$$(\mathrm{d}x^\mu\, \partial_\mu + \mathrm{d}\theta\, \delta + \mathrm{d}\bar{\theta}\, \bar{\delta})\hat{\mathcal{F}} + [\hat{\mathcal{A}}, \hat{\mathcal{F}}] = 0 \tag{15}$$

guarantees that the powers of $\hat{\mathcal{F}}$ obey

$$(\mathrm{d}x^\mu\, \partial_\mu + \mathrm{d}\theta\, \delta + \mathrm{d}\bar{\theta}\, \bar{\delta})\, \mathrm{Tr}\, \hat{\mathcal{F}}^n = 0. \tag{16}$$

The simplest one is the superspace 4-form $\hat{\mathcal{F}}^2$. It can be expanded in powers of $\mathrm{d}\theta$ and $\mathrm{d}\bar{\theta}$:

$$\tfrac{1}{2}\, \mathrm{Tr}\, \hat{\mathcal{F}}^2 = \sum_{i,j;i+j\leqslant 4} W^{i,j} \mathrm{d}\theta^i \mathrm{d}\bar{\theta}^j. \tag{17}$$

Equation (16) gives

$$\mathrm{d}W^{i,j} + \delta W^{i-1,j} + \bar{\delta} W^{i,j-1} = 0. \tag{18}$$

When $j = 0$ the integrals of the $(4-i)$-form $W^{i,0}$ over a $(4-i)$ cocycle γ are BRST closed:

$$\int_\gamma \mathrm{d}W^{i,0} + \delta \int_\gamma W^{i-1,0} = \int_{\partial\gamma} W^{i,0} + \delta \int_\gamma W^{i-1,0} = \delta \int_\gamma W^{i-1,0} = 0. \tag{19}$$

Because of (18) the BRST cohomology of $\int W$ depends only on the homology class of γ, making the vacuum expectation values and correlation functions of $\int W$ good candidates for topological invariants. Note that because of the symmetry of the three-dimensional action (4) and (10) under the partial integration transformation the expectation values of any observable $\mathcal{O}$ and its 'conjugate' $\overline{\mathcal{O}}$ are the same. This can be seen by making a change of variables (with a unit Jacobian) in the path integral taking all the fields to their conjugates and using the invariance of the action. The condition $\delta\mathcal{O} = 0$ changes under this transformation to $\bar{\delta}\overline{\mathcal{O}} = 0$. Therefore, objects that are either δ- or $\bar{\delta}$-closed qualify as observables of $3d$ SBF. In particular, we can thus identify $\overline{W}^{i,j} = (-1)^j W^{j,i}$.

The expansion of $\hat{\mathcal{F}}^2$ gives using (6)

$$\begin{aligned}
W^{00} &= \tfrac{1}{2}F^2\\
W^{10} &= \psi F - \bar{\theta}(BF - \chi\,\mathrm{d}_A\psi)\\
W^{20} &= \tfrac{1}{2}\psi^2 + \phi F + \theta(\phi\,\mathrm{d}_A\psi) - \bar{\theta}(\psi B + \phi\,\mathrm{d}_A\chi + F\eta) + \theta\bar{\theta}(\phi\,\mathrm{d}_A B - \phi[\psi,\chi] + \mathrm{d}_A\psi\eta)\\
W^{30} &= \psi\phi - \bar{\theta}(\phi B + \psi\eta)\\
W^{40} &= \tfrac{1}{2}\phi^2 - \bar{\theta}(\phi\eta)
\end{aligned}\tag{20}$$

from which we can extract 11 observables. The previously unknown ones are the θ and $\theta\bar{\theta}$, components of W^{20}. They are particular to three-dimensional theories and they are unlike the others, which, or rather their generalizations involving all the Batalin–Vilkovisky ghosts, can be obtained from the truncated supercurvatures $\hat{\mathcal{F}}$ and $\hat{\mathcal{R}}$ of A and B sectors in all dimensions. Nevertheless, the θ component of W^{20} seems to be BRST closed also in higher dimensions: the ghosts for ghosts and other fields appear only in the transformations for the B sector.

Interestingly, some of the observables above are formally the same as for Donaldson theory. This is no surprise since the BRST structure of Donaldson theory is similar to that of the A sector of the SBF. In fact, the SBF can be thought of as a reduction of the Donaldson theory to three dimensions [13, 16].

As a characteristic for the Witten-type topological theories, the expectation values of the observables

$$\langle\mathcal{O}\rangle = \int [\mathrm{d}X]\mathcal{O}\,\mathrm{e}^{\mathrm{i}/g^2 S_{\mathrm{q}}} \tag{21}$$

are independent of the coupling g^2. The integral can be calculated in the $g^2 \to 0$ limit where it localizes to the classical equations of motion

$$F_A = 0 \qquad \mathrm{d}_A\psi = 0 \qquad \mathrm{d}_A B - [\psi,\chi] = 0 \qquad \mathrm{d}_A\chi = 0 \tag{22}$$

i.e. it is now calculated over the moduli space of flat connections $\mathcal{M}$. In the limit $g^2 \to 0$ the fields in (20) are replaced by their classical values (22). Then the non-vanishing observables are

$$\begin{aligned}
&\omega_0^4 = \tfrac{1}{2}\phi^2 \qquad &&\omega_1^3 = \int\psi\phi \qquad &&\omega_2^2 = \tfrac{1}{2}\int\psi^2\\
&\omega_0^3 = \phi\eta \qquad &&\omega_1^2 = \int\psi\eta + B\phi \qquad &&\omega_2^1 = \int\psi B.
\end{aligned}\tag{23}$$

To evaluate the expectation values one has to take care of the zero modes of the fermions. In particular, in dimensions higher than two the zero-modes of the other fields complicate matters considerably. We will not perform the calculations here but refer the reader to [12] and references therein for discussions on topological invariants of the Donaldson theory. The considerations there are quite similar to those for the observables ω_0^4, ω_1^3 and ω_2^2 of the A sector of SBF. The invariant corresponding to ω_2^2 has been evaluated in [12] for $2d$ BF and found to be the symplectic volume of the moduli space. Its products with ω_0^4 produce linking and intersection numbers of moduli spaces.

4. Vector supersymmetry and the tower of observables

A peculiar feature of Chern–Simons and BF theories is a vector-like supersymmetry of the action [11]. This gives rise to new Ward identities which have been utilized in proving the theories to be finite, renormalizable and free of anomalies [10, 11, 19–21]. This supersymmetry depends explicitly on the metric so it is not clear whether it can be used if the space is not flat. However, the finiteness of the CS theory on asymptotically free manifolds has been proven in [22] by using a local version of the vector supersymmetry. Also because the theories are topological one might argue that their physical quantities are not dependent on the metric of the manifold. In any case, the vector supersymmetry has been established as a common feature of many topological theories [9] and a useful tool, not only in the study of renormalization and related topics but also in finding new observables.

The vector supersymmetry for non-supersymmetric CS theory quantized in the Landau gauge is generated by an operator s with ghost number and form degree 1,

$$\begin{aligned} &s\mathcal{A} = *\mathrm{d}\mathcal{C} && s\mathcal{C} = 0 \\ &s\bar{\mathcal{C}} = \mathcal{A} && s\Lambda = -\delta\mathcal{A} \end{aligned} \tag{24}$$

or written in the component form

$$s = s_\alpha\,\mathrm{d}x^\alpha \qquad *\,\mathrm{d}\mathcal{C} = -\epsilon_{\mu\alpha\beta}\partial^\beta\mathcal{C}\,\mathrm{d}x^\mu\,\mathrm{d}x^\alpha.$$

Using the partial integration for the CS theory one can obtain the anti-supersymmetry $\bar{s}$:

$$\begin{aligned} &\bar{s}\mathcal{A} = *\mathrm{d}\bar{\mathcal{C}} && \bar{s}\bar{\mathcal{C}} = 0 \\ &\bar{s}\mathcal{C} = -\mathcal{A} && \bar{s}\Lambda = -\bar{\delta}\mathcal{A} - [\mathcal{A}, \bar{\mathcal{C}}]. \end{aligned} \tag{25}$$

The anticommutation relations of the operators $\delta, \bar{\delta}$ and $s, \bar{s}$ are

$$[s_\alpha, s_\beta] = [\bar{s}_\alpha, \bar{s}_\beta] = [\delta, \bar{\delta}] = [s_\alpha, \bar{s}_\beta] = [\delta, s_\alpha] = [\bar{\delta}, \bar{s}_\alpha] = 0 \tag{26}$$

$$[\delta, \bar{s}_\alpha] = -[\bar{\delta}, s_\alpha] = \partial_\alpha + \text{terms vanishing modulo the equations of motion.}$$

Together with the BRST operators δ and $\bar{\delta}$, the operators s and $\bar{s}$ can be combined to form a generator of $N = 2$ supersymmetry algebra [9, 11, 17]. The vector supersymmetries can also be formulated for the non-supersymmetric BF theories. In dimensions other than three there exists no vector supersymmetry s but the $\bar{s}$ operator can still be constructed [10, 20].

The anti-vector supersymmetry can be generalized to the supersymmetric BF theory in arbitrary dimensions. In 3d it can be derived easily using (25) for the superfields $\hat{A}^0$, $\hat{A}^1_\theta$, $\hat{A}^{-1}_{\bar{\theta}}$ and $\hat{\Lambda}$:

$$\begin{aligned} &\bar{s}_\alpha A_\mu = -\epsilon_{\mu\alpha\beta}\partial^\beta\bar{\eta} && \bar{s}_\alpha B = -\epsilon_{\mu\alpha\beta}\partial^\beta\bar{c} \\ &\bar{s}_\alpha c = A_\alpha && \bar{s}_\alpha\eta = B_\alpha \\ &\bar{s}_\alpha\bar{c} = 0 && \bar{s}_\alpha\bar{\eta} = 0 \\ &\bar{s}_\alpha b = -\partial_\alpha\bar{c} && \bar{s}_\alpha\lambda = D_\alpha\bar{\eta} \\ &\bar{s}_\alpha\psi_\mu = \epsilon_{\mu\alpha\beta}\partial^\beta\bar{\rho} && \bar{s}_\alpha\chi = \epsilon_{\mu\alpha\beta}\partial^\beta\bar{\phi} \\ &\bar{s}_\alpha\phi = -\psi_\alpha && \bar{s}_\alpha\rho = -\chi_\alpha \\ &\bar{s}_\alpha\bar{\phi} = 0 && \bar{s}_\alpha\bar{\rho} = 0 \\ &\bar{s}_\alpha\beta = \partial_\alpha\bar{\phi} && \bar{s}_\alpha\sigma = D_\alpha\bar{\rho}. \end{aligned} \tag{27}$$

This is a symmetry of the quantum action (4) and (10) and satisfies the anticommutation relations (26) with the BRST operator (7) and (9). The analysis performed on the renormalization, finiteness and anomalies of ordinary BF theories using vector supersymmetry can thus be applied directly to the supersymmetric BF theories.

It is interesting to note that the vector supersymmetry of the SBF can also be useful in constructing new observables (see [9] for a slightly different approach). Whenever there exists a BRST closed object ω, $\bar{s}\omega$ is also BRST closed as a result of the anticommutation relations (26). So in principle it is possible to find an observable, like ω_0^4 and ω_0^3, and apply $\bar{s}_\alpha$ successively to obtain new ones. Also, since

$$\omega_0^4 = \tfrac{1}{2}\delta(c\phi - \tfrac{1}{6}c[c, c]) \qquad \omega_0^3 = \tfrac{1}{2}\delta(\phi\rho + c\eta - \tfrac{1}{2}\rho[c, c]) = \delta(\phi\rho) \tag{28}$$

all observables obtained by acting with $\bar{s}$ are in fact BRST exact—modulo equations of motion and surface terms. This is valid only locally and does not mean that the observables should be trivial.

From (20) we see that modifying slightly the antisupersymmetry transformations for ψ and B in (27) as

$$\bar{s}B = *\mathrm{d}\bar{c} - 2\mathrm{d}_A\chi \qquad \bar{s}\psi = - * \mathrm{d}\bar{\rho} - 2F$$

and leaving the others intact the antisupersymmetry still remains a symmetry of the action. By denoting the metric independent part of the modified $\bar{s}$ operator by $\bar{v}$,

$$\begin{aligned} &\bar{v}B = -2\mathrm{d}_A\chi \qquad &&\bar{v}\eta = -B \qquad &&\bar{v}\rho = -\chi \\ &\bar{v}\psi = -2F \qquad &&\bar{v}c = -A \qquad &&\bar{v}\phi = -\psi \end{aligned} \tag{29}$$

and applying successively $(1/k!)(-\bar{v})^k$ to $\frac{1}{2}\phi^2$ and $\phi\eta$ it is possible to derive all the observables in (20), except the θ and $\theta\bar{\theta}$ components of W^{02}. The symmetry $\bar{v}$ acts as a vertical (in the direction of the form degree) transformation along the components W^{ij} of the supersurvature (5).

It is easily seen that a horizontal (ghost number direction) transformation $\bar{h}$ can also be defined:

$$\begin{aligned} &\bar{h}A = \chi \qquad &&\bar{h}c = -\rho \\ &\bar{h}\psi = -B \qquad &&\bar{h}\phi = -\eta \\ &\bar{h}\bar{\eta} = -\bar{\phi} \qquad &&\bar{h}\bar{\rho} = -\bar{c}. \end{aligned} \tag{30}$$

This is a symmetry of the action and commutes with the BRST operator. It thus allows us to construct all possible observables starting from the element $\frac{1}{2}\phi^2$ of highest ghost number and lowest form degree—again excluding the θ and $\theta\bar{\theta}$ components of W^{20}.

The vertical transformation has geometrical interpretation as the equivariant derivative of the BRST model acting on the curvature of the universal bundle over the space of gauge connections [23], which can be identified with the supercurvature $\hat{F}$. Therefore, the vertical transformation can be defined for all Witten-type topological theories. The horizontal transformation $\bar{h}$ which can be constructed only in three dimensions is in fact part of the anti-BRST operator $-\delta$: only those terms that are not composites of fields and do not contain Lagrange multipliers are included.

Acting on ϕ^2 with the total vector supersymmetry transformation $\bar{s}$ instead of the vertical transformation we obtain an even larger set of observables. In addition to those present in (20) these include observables that depend explicitly on the metric. Since we already know that the observables in (20) are BRST closed, the metric dependent ones should also be closed separately. Moreover, it can be shown that the metric variations of these observables can be written as BRST exact terms, a necessary requirement for the observables to be topological invariants [12]. So we can conclude that the expectation values of these observables are of topological nature.

5. Conclusions

We have studied three-dimensional supersymmetric BF theories using the superspace formalism. This has proved to be a powerful method for studying the properties of the theory and especially for finding new symmetries and observables. The superspace curvature gives rise to a hierarchy of observables, which could be derived starting from one initial observable using the two transformations we constructed. The transformations have a geometrical interpretation as vertical and horizontal transformations acting on the components of the supercurvature, and can be identified as parts of more general symmetries of the action, the vector supersymmetry and anti-BRST symmetry.

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